Optimization with constraints (multiple optimal points)

The objective function is:

$$f(x, y) = x - y^2 + 1$$

The constraint:

$$(x - 1)^2 + y^2 = 1$$

Find the optimal points and prove whether they are minima or maxima using the second-order conditions.

Solution

We formulate the Lagrangian

$$\mathcal{L} = x - y^2 + 2 + \lambda\left[1 - (x-1)^2 - y^2\right]$$

We calculate the first-order conditions:

$$\mathcal{L} = x - y^2 + 2 + \lambda\left[1 - (x^2 - 2x + 1) - y^2\right]$$
$$\mathcal{L} = x - y^2 + 2 + \lambda\left[1 - x^2 + 2x - 1 - y^2\right]$$
$$\mathcal{L} = x - y^2 + 2 + \lambda\left[-x^2 + 2x - y^2\right]$$

$$\frac{\partial \mathcal{L}}{\partial x} = 1 - \lambda 2x + 2\lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial y} = -2y - 2y\lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = -x^2 + 2x - y^2 = 0$$

From the first condition:

$$1 = \lambda(2x - 2)$$

If we assume that $x \neq 1$

$$\frac{1}{2x-2} = \lambda$$

We insert this into the second equation:

$$y + y\left(\frac{1}{2x-2}\right) = 0$$
$$y = -\frac{y}{2x-2} \Rightarrow (2x-2)y = -y$$
$$2xy - 2y = -y \Rightarrow 2xy = y$$

If we assume that $y \neq 0$

$$x = 1/2$$

We insert this into the third equation:

$$-x^2 + 2x - y^2 = 0$$
$$-\left(\frac{1}{2}\right)^2 + 2\frac{1}{2} - y^2 = 0$$
$$-\frac{1}{4} + 1 - y^2 = 0$$
$$\frac{3}{4} = y^2 \Rightarrow y = \pm\sqrt{\frac{3}{4}}$$
$$y = \pm\frac{\sqrt{3}}{2}$$

The associated value of λ would be:

$$\lambda = \frac{1}{2\frac{1}{2} - 2} = -1$$

So we have 2 points:

$$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, -1\right), \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}, -1\right)$$

If $y = 0$, then from the third equation:

$$-x^2 + 2x = 0$$
$$x(-x + 2) = 0$$

From here we get that $x = 2$ or $x = 0$. This is associated with 2 values of λ:

$$\lambda = 1/2$$

Or:

$$\lambda = -1/2$$

This leads us to two more points:

$$(2, 0, 1/2), (0, 0, -1/2)$$

Lastly, if we assume that $x = 1$, the first equation does not hold:

$$1 - \lambda 2x + 2\lambda = 0$$

$$1 = 0$$

We are left with the 4 points found and move on to verify the second-order conditions:

$$\mathcal{L}_{xx} = -2\lambda$$
$$\mathcal{L}_{yy} = -2 - 2\lambda$$
$$\mathcal{L}_{xy} = \mathcal{L}_{yx} = 0$$
$$g'_x = -2x + 2$$
$$g'_y = -2y$$

We formulate the bordered Hessian:

$$\bar{H} = \begin{pmatrix} 0 & -2x+2 & -2y \\ -2x+2 & -2\lambda & 0 \\ -2y & 0 & -2-2\lambda \end{pmatrix}$$

We calculate the determinant:

$$|\bar{H}| = (2x - 2)[(-2x + 2)(-2 - 2\lambda)] - 2y[-4\lambda y] = (2x - 2)[4x + 4x\lambda - 4 - 4\lambda] + 8\lambda y^2$$

$$|\bar{H}| = 2(x - 1)[2x + 2x\lambda - 2 - 2\lambda] + 8\lambda y^2$$

We evaluate each of the points in the determinant of the bordered Hessian:

$$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, -1\right) \Rightarrow |\bar{H}| = -6 < 0 \quad \text{min}$$
$$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}, -1\right) \Rightarrow |\bar{H}| = -6 < 0 \quad \text{min}$$
$$(0, 0, -1/2) \Rightarrow |\bar{H}| = 2 > 0 \quad \text{max}$$
$$(2, 0, 1/2) \Rightarrow |\bar{H}| = 6 > 0 \quad \text{max}$$